Junior Mathematical Challenge

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Solutions and investigations

April 2022

These solutions augment the shorter solutions also available online. The solutions given here are full solutions, as explained below. In some cases we give alternative solutions. There are also many additional problems for further investigation. We welcome comments on these solutions. Please send them to challenges@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that occasionally you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can sometimes be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, sometimes, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).

Enquiries about the Junior Mathematical Challenge should be sent to:

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	A	D	D	C	B	D	E	B	D	E	B	D	B	E	C	C	E	B	B	C	E	D	A	C

1. Which of these has the greatest value?

A $20 + 22$ B $202 + 2$ C 202×2 D $2 \times 0 \times 2 \times 2$ E 20×22

Solution **E**

We have

$$\begin{aligned} 20 + 22 &= 42, \\ 202 + 2 &= 204, \\ 202 \times 2 &= 404, \\ 2 \times 0 \times 2 \times 2 &= 0 \\ \text{and} \quad 20 \times 22 &= 440. \end{aligned}$$

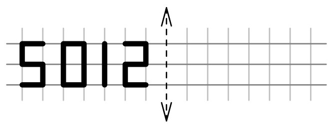
From this we see that 20×22 has the greatest value.

For investigation

1.1 Which of these has the least value?

(a) $20 - 22$ (b) $202 - 2$ (c) $202 \div 2$ (d) $2 \times 0 - 2 \times 2$ (e) $20 \div 22$

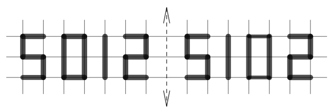
2. The number 5012 is reflected in the mirror-line shown. Onto which number is it reflected?



A 5102 B 2015 C 5012 D 2105 E 5105

Solution **A**

The digit 2 reflects to 5, and 5 reflects to 2. Also, 0 reflects to 0 and 1 reflects to 1.



Therefore, the number 5012 reflects to 5102.

For investigation

2.1 We have noted in the above solution that the digits 2 and 5 are interchanged when they are reflected, and the digits 0 and 1 are unchanged. The digit 8 is also unchanged when reflected in the mirror line shown in the question. None of the other digits, 3, 4, 6, 7 and 9, reflect to another digit.

Thus five digits reflect to a digit, and five do not.

(a) How many two-digit numbers reflect to a two-digit number?

(b) How many three-digit numbers reflect to a three-digit number?

(c) How many four-digit numbers reflect to a four-digit number?

(d) How many four-digit numbers are unchanged by reflection in the mirror line shown in the question?

3. Think of any number. Add five; multiply by two; add ten; divide by two; subtract your original number; add three.
What is the resulting number?

A 10 B 11 C 12 D 13 E 14

SOLUTION **D**

The wording of the question implies that the resulting number will be the same whatever number you think of. Therefore, a straightforward way to answer the question is to think of a particular number and work out the resulting number.

This is what happens if the number I think of is 1:

$$1 \longrightarrow \boxed{+5} \xrightarrow{6} \boxed{\times 2} \xrightarrow{12} \boxed{+10} \xrightarrow{22} \boxed{\div 2} \xrightarrow{11} \boxed{-x} \xrightarrow{10} \boxed{+3} \longrightarrow 13$$

In the context of the JMC we may conclude that the resulting number is 13 whatever number you think of.

Note: Of course, checking that for the input 1, the output is 13 goes no way to *proving* that, whatever the input number the output number is bound to be 13. To adapt a comment of the late John Conway, when you have checked that the output is 13 for a million different input values, how many values do there remain to be checked?

The answer is that there still remain infinitely many values to be checked.

This shows that the claim that the output is 13 for all input values cannot be proved by checking cases. What is needed is a general argument, using algebra. You are asked to provide this in Problem 3.1.

FOR INVESTIGATION

3.1 Check that whichever number you think of, the resulting number is 13. That is, show that if the number you think of is x, the resulting number is 13 independently of the value of x.

3.2 Check that for each number x the sequence of operations shown in this diagram

$$x \longrightarrow \boxed{+4} \longrightarrow \boxed{\times 6} \longrightarrow \boxed{-39} \longrightarrow \boxed{\div 3} \longrightarrow \boxed{-x} \longrightarrow \boxed{+5} \longrightarrow x$$

always produces the output x.

3.3 Find positive integers P, Q, R, S and T so that for each number x the sequence of operations shown in this diagram

$$x \longrightarrow \boxed{+P} \longrightarrow \boxed{\times Q} \longrightarrow \boxed{-R} \longrightarrow \boxed{-x} \longrightarrow \boxed{\div S} \longrightarrow \boxed{+T} \longrightarrow x$$

always produces the output x.

4. What is the value of $0.6 + \frac{2}{5}$?

A 0.15 B 0.24 C 0.8 D 1 E 2.4

SOLUTION **D**

Note: We can evaluate the sum using either decimals or fractions.

In the first method we use the fact that we can express the fraction $\frac{2}{5}$ as the decimal 0.4.

In the second method we use the fact that we can express the decimal 0.6 as the fraction $\frac{3}{5}$.

METHOD 1

$$0.6 + \frac{2}{5} = 0.6 + 0.4 = 1.$$

METHOD 2

$$0.6 + \frac{2}{5} = \frac{3}{5} + \frac{2}{5} = \frac{5}{5} = 1.$$

FOR INVESTIGATION

4.1 Evaluate each of the following expressions giving your answers both as fractions and as decimals.

(a) $0.75 + \frac{3}{8}$.

(b) $0.4 + \frac{2}{3}$.

(c) $0.\dot{3} + \frac{1}{7}$.

4.2 Evaluate each of the following expressions giving your answers both as fractions and as decimals.

(a) $0.75 \times \frac{3}{8}$.

(b) $0.4 \times \frac{2}{3}$.

(c) $0.\dot{3} \times \frac{1}{7}$.

Note: In Problems 4.1 (c) and 4.2 (c) the notation $0.\dot{3}$ stands for the recurring decimal $0.333333\ldots$ which represents the fraction $\frac{1}{3}$.

5. How many of the following take integer values?

$$\frac{1}{1} \qquad \frac{11}{1+1} \qquad \frac{111}{1+1+1} \qquad \frac{1111}{1+1+1+1} \qquad \frac{11\,111}{1+1+1+1+1}$$

A 0 B 1 C 2 D 3 E 4

SOLUTION **C**

Note: We need to test each of the options in turn. To decide whether a fraction $\frac{p}{q}$ has an integer value it is not necessary to work out the value of $p \div q$. Instead all we need do is decide whether p is divisible by q. Often we can do this by using standard divisibility tests.

A Every integer is divisible by 1. Therefore $\frac{1}{1}$ is an integer.

B Since 11 is an odd number, it is not divisible by 2. Therefore $\frac{11}{1+1} = \frac{11}{2}$ is not an integer.

C An integer is divisible by 3 if the sum of its digits is divisible by 3. Since $1 + 1 + 1 = 3$, it follows that 111 is divisible by 3. Therefore $\frac{111}{1+1+1} = \frac{111}{3}$ is an integer.

D An integer is divisible by 4 only if its last two digits form a number which is divisible by 4. Since 11 is not divisible by 4, neither is 1111. Therefore $\frac{1111}{1+1+1+1} = \frac{1111}{4}$ is not an integer.

E An integer is divisible by 5 only if its last digit is 0 or 5. Therefore 11 111 is not divisible by 5. Hence $\frac{11111}{1+1+1+1+1} = \frac{11111}{5}$ is not an integer.

Therefore just 2 of the expressions have values that are integers.

FOR INVESTIGATION

5.1 Check that $111 = 3 \times 37$. (This is a numerical fact that is worth remembering.)

5.2 In the solution above we used the following tests for divisibility. Explain why each of them is correct.

(a) The test for an integer to be divisible by 3 is that the sum of its digits should be divisible by 3.

(b) The test for an integer to be divisible by 4 is that its last two digits should form a number that is divisible by 4.

(c) The test for an integer to be divisible 5 is that its last digit should be 0 or 5.

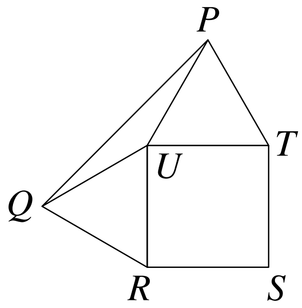
5.3 Find similar tests for divisibility by 8, by 9, by 10 and by 11. [There is no very simple test for divisibility by 7.] Here "find" means either discover for yourself, or look in a book, or search the internet, or ask your teacher.

5.4 Which of the numbers 2, 3, 4, 5, 6, 8, 9, 10 and 11 is the number 123 456 789 987 654 321 divisible by?

6. The diagram shows the square $RSTU$ and two equilateral triangles, PUT and QRU.

What is the size of angle QPU?

A $10°$ B $15°$ C $20°$ D $25°$ E $30°$



Solution **B**

The angles in a square are each $90°$. The angles in an equilateral triangle are each $60°$. The angles at a point total $360°$.

Therefore, by considering the angles at the point U, we have

$$\angle PUQ = 360° - 60° - 90° - 60° = 150°.$$

Therefore, because the sum of the angles in a triangle is $180°$,

$$\angle QPU + \angle PQU = 180° - 150° = 30°. \qquad (1)$$

We also have $QU = UR = UT = PU$. Therefore the triangle PUQ is isosceles with $PU = QU$.

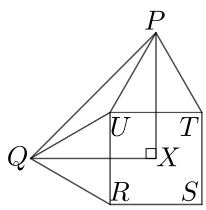
Hence $$\angle PQU = \angle QPU. \qquad (2)$$

We deduce from (1) and (2) that $2\angle QPU = 30°$. Therefore $\angle QPU = 15°$.

For investigation

6.1 Suppose that the square and the equilateral triangles in this question have side length 1. What is the length of QP?

[Hint: Let X be the centre of the square. Find the lengths of PX and QX. Now apply Pythagoras' theorem to the right-angled triangle PXQ.]



6.2 We again suppose that the square and the equilateral triangles in this question have side length 1.

Then in the triangle PQU we have $PU = QU = 1$ and $\angle QPU = \angle PQU = 15°$.

Now use the answer to Problem 6.1, to obtain a formula for $\cos 15°$ in terms of $\sqrt{2}$ and $\sqrt{6}$.

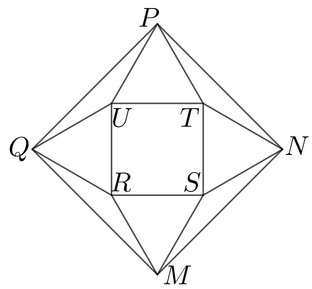
[Hint: Let Y be the midpoint of PQ. Show that the triangles PUY and QUY are congruent. Deduce that $\angle UYP = \angle UYQ = 90°$.]

6.3 This problem provides an alternative method for answering Question 6.

Construct equilateral triangles MSR and NTS on the other two sides of the square, as shown in the diagram.

Use the symmetry of the figure to show that $MNPQ$ is a square.

Deduce that $\angle QPU = 15°$.



7. Kiwi fruit contain roughly two and a half times as much vitamin C as the same weight of oranges.
What weight of kiwi fruit contains approximately the same amount of vitamin C as 1 kg of oranges?

A 100 g B 200 g C 250 g D 400 g E 550 g

Solution **D**

Weight for weight, kiwi fruit contains approximately $2\frac{1}{2}$ times as much vitamin C as oranges. Therefore the weight of kiwi fruit containing approximately the same amount of vitamin C as 1 kg of oranges, is

$$\left(\frac{1}{2\frac{1}{2}}\right)\text{kg}.$$

Now

$$\left(\frac{1}{2\frac{1}{2}}\right) = \left(\frac{1}{\frac{5}{2}}\right) = \frac{2}{5}.$$

Therefore, as $1\,\text{kg} = 1000\,\text{g}$,

$$\left(\frac{1}{2\frac{1}{2}}\right)\text{kg} = \frac{2}{5} \times 1000\,\text{g} = 400\,\text{g}.$$

For investigation

7.1 What weight of kiwi fruit contains approximately the same amount of vitamin C as 475 g of oranges?

7.2 What weight of oranges contains approximately the same amount of vitamin C as 1 kg of kiwi fruit?

8. Today is Thursday. What day will it be in 100 days' time?

A Tuesday B Wednesday C Thursday D Friday E Saturday

Solution **E**

$100 = 14 \times 7 + 2$. Therefore 100 days is 14 weeks and two days. Two days after a Thursday it is a Saturday. Therefore 100 days after a Thursday it is also a Saturday.

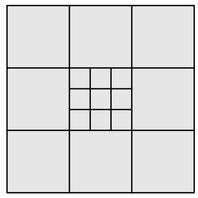
For investigation

8.1 Which day of the week is it 200 days after a Thursday?

8.2 Which day of the week is it 1001 days after a Thursday?

9. How many squares of any size can be seen in the diagram?

A 25 B 27 C 28 D 39 E 40

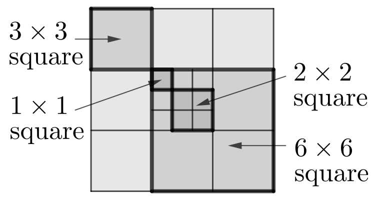


Solution **B**

So that we can easily refer to the differently sized squares in the diagram, we suppose that the smallest squares in the centre of the diagram each have size 1×1. It follows that the largest square has size 9×9.

We now see that there are

9 squares of size 1×1,
4 squares of size 2×2,
9 squares of size 3×3,
4 squares of size 6×6,
and 1 square of size 9×9.



This makes a total of 27 squares.

10. Half of a quarter of an eighth of a number is equal to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.
What is the number?

A 14 B 28 C 42 D 56 E 64

Solution **D**

Let the number be x. Then

$$\frac{1}{2}\left(\frac{1}{4}\left(\frac{1}{8}x\right)\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

That is,
$$\frac{1}{64}x = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}.$$

Therefore
$$x = 64 \times \frac{7}{8} = 8 \times 7 = 56.$$

For investigation

10.1 Half of a quarter of an eighth of a sixteenth of a number is equal to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$. What is the number?

10.2 One third of a ninth of a twenty-seventh of a number is equal to $\frac{1}{3} + \frac{1}{9} + \frac{1}{27}$. What is the number?

11. Nine of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are to be put in two groups so that the sum of the numbers in each group is a multiple of four.
What is the largest number that could be left out?

A 3 B 4 C 5 D 6 E 7

Solution **E**

For the nine numbers to be put in two groups so that the sum of the numbers in each group is a multiple of 4, the sum of all the nine numbers needs to be a multiple of 4.

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$. We have $55 = 4 \times 13 + 3$ and so 55 is 3 more than a multiple of 4.

To make the total a multiple of 4 we need to leave out a number that has remainder 3 when divided by 4. Therefore the largest number that could be left out is 7. Then the sum of the remaining numbers is $1 + 2 + 3 + 4 + 5 + 6 + 8 + 9 + 10 = 48$, and so it is a multiple of 4.

To be sure that this is the correct answer, we need to check that the remaining numbers when 7 is removed may be put in two groups so that the sum of the numbers in each of the groups is a multiple of 4. You are asked to do this in Problem 11.1.

For investigation

11.1 Show how the numbers 1, 2, 3, 4, 5, 6, 8, 9, 10 can be put into two groups so that the sum of the numbers in each group is a multiple of 4.

11.2 Show how the numbers, 1, 2, 3, 4, 5, 6, 8, 9, 10 can be put into four groups so that the sum of the numbers in each group is a multiple of 4.

12. When my pot of paint is half full, it weighs 5.8 kg. When my pot of paint is one quarter full, it weighs 3.1 kg. What is the weight of the full pot?

A 8.9 kg B 11.2 kg C 11.6 kg D 12 kg E 12.4 kg

Solution **B**

The difference between the weight of the half full pot and the weight of the quarter full pot is the weight of the amount of paint that fills one quarter of the pot. Therefore this weight is $5.8\,\text{kg} - 3.1\,\text{kg} = 2.7\,\text{kg}$.

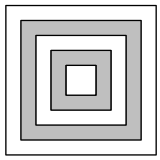
Therefore the amount of paint that half fills the pot is $2 \times 2.7\,\text{kg} = 5.4\,\text{kg}$.

The half full pot weights 5.8 kg. To make a full pot we would need to add the amount of paint that half fills the pot. We have seen that this amount of paint weighs 5.4 kg. Therefore the full pot weighs $5.8\,\text{kg} + 5.4\,\text{kg} = 11.2\,\text{kg}$.

For investigation

12.1 What fraction of the pot is filled with paint when the weight of the paint in the pot is nine times the weight of the empty pot?

13. The diagram shows five squares whose side-lengths, in cm, are 1, 2, 3, 4 and 5.



What percentage of the area of the outer square is shaded?

A 25% B 30% C 36% D 40% E 42%

Solution **D**

The area of the outer square with side length 5 cm is $5^2\,\text{cm}^2 = 25\,\text{cm}^2$.

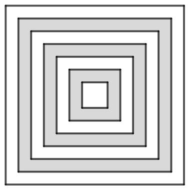
The area of the shaded region between the square with side length 4 cm and the square with side length 3 cm is $4^2\,\text{cm}^2 - 3^2\,\text{cm}^2 = 16\,\text{cm}^2 - 9\,\text{cm}^2 = 7\,\text{cm}^2$. The area of the shaded region between the square with side length 2 cm and the square with side length 1 cm is $2^2\,\text{cm}^2 - 1^2\,\text{cm}^2 = 4\,\text{cm}^2 - 1\,\text{cm}^2 = 3\,\text{cm}^2$.

Therefore the total shaded area is $7\,\text{cm}^2 + 3\,\text{cm}^2 = 10\,\text{cm}^2$.

It follows that the fraction of the diagram that is shaded is $\dfrac{10}{25} = \dfrac{2}{5}$. Expressed as a percentage this is 40%.

For investigation

13.1 The diagram on the right shows seven squares whose side lengths, in cm, are 1, 2, 3, 4, 5, 6 and 7. What fraction of the area of the outer square is shaded?



13.2 Find a formula, in terms of n, for the fraction of the area of the outer square that is shaded in the general case with $2n + 1$ squares with side lengths, in cm, $1, 2, 3, \ldots, 2n$ and $2n + 1$.

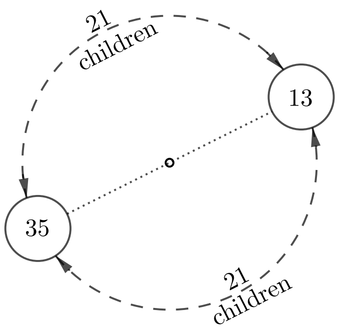
14. A group of children stand evenly spaced around a circular ring and are numbered consecutively 1, 2, 3, and so on. Number 13 is directly opposite number 35.

How many children are there in the ring?

A 42 B 44 C 46 D 48 E 50

Solution **B**

Because child number 13 is directly opposite child number 35, there are the same number of children going anti-clockwise between them as there are going clockwise.



Between child number 13 and child number 35 going clockwise there are the children with numbers from 14 to 34, inclusive. That makes 21 children.

Therefore there are also 21 children between child number 13 and child number 35 going anticlockwise. Including child numbers 13 and 35 this makes a total of $21 + 21 + 2 = 44$ children.

15. What is the value of $2 \div (4 \div (6 \div (8 \div 10)))$?

A $\frac{1}{960}$ B $\frac{1}{5}$ C $\frac{3}{8}$ D $\frac{1}{2}$ E $\frac{15}{4}$

Solution **E**

We have

$$\begin{aligned}
2 \div (4 \div (6 \div (8 \div 10))) &= 2 \div \left(4 \div \left(6 \div \frac{8}{10}\right)\right) \\
&= 2 \div \left(4 \div \left(6 \times \frac{10}{8}\right)\right) \\
&= 2 \div \left(4 \div \frac{6 \times 10}{8}\right) \\
&= 2 \div \left(4 \times \frac{8}{6 \times 10}\right) \\
&= 2 \div \frac{4 \times 8}{6 \times 10} \\
&= 2 \times \frac{6 \times 10}{4 \times 8} \\
&= \frac{2 \times 6 \times 10}{4 \times 8} \\
&= \frac{15}{4}.
\end{aligned}$$

For investigation

15.1 Find the value of

(a) $2 \div (4 \div (6 \div (8 \div (10 \div 12))))$.

(b) $2 \div (4 \div (6 \div (8 \div (10 \div (12 \div 14)))))$.

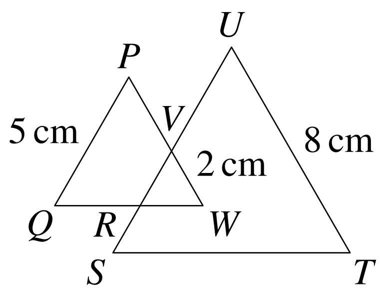
15.2 Can you generalize these answers? That is, can you find formulas in terms of n for the values of

(i) $2 \div (4 \div (6 \div (\ldots ((4n - 2) \div 4n) \ldots)))$,

and

(ii) $2 \div (4 \div (6 \div (\ldots (4n \div (4n + 2)) \ldots)))$?

16. The diagram shows a seven-sided polygon, $PQRSTUV$. It is formed by two equilateral triangles PQW and STU of side-length 5 cm and 8 cm respectively.
The two triangles overlap in an equilateral triangle of side-length 2 cm.



What is the perimeter of $PQRSTUV$?

A 27 cm B 30 cm C 33 cm D 36 cm E 39 cm

SOLUTION **C**

Because the triangles PQW, STU and RWV are equilateral triangles with side lengths 5 cm, 8 cm and 2 cm, respectively,

$$PW = QW = PQ = 5\,\text{cm},$$
$$ST = TU = US = 8\,\text{cm}$$

and

$$RW = VW = VR = 2\,\text{cm}.$$

Hence

$$QR = QW - RW = 5\,\text{cm} - 2\,\text{cm} = 3\,\text{cm}.$$

Similarly

$$VP = PW - VW = 5\,\text{cm} - 2\,\text{cm} = 3\,\text{cm}.$$

Also,

$$RS + UV = US - VR = 8\,\text{cm} - 2\,\text{cm} = 6\,\text{cm}.$$

Therefore the perimeter of the polygon $PQRSTUV$ is

$$\begin{aligned} &PQ + QR + RS + ST + TU + UV + VP \\ &\qquad = PQ + QR + (RS + UV) + ST + TU + VP \\ &\qquad = 5\,\text{cm} + 3\,\text{cm} + 6\,\text{cm} + 8\,\text{cm} + 8\,\text{cm} + 3\,\text{cm} \\ &\qquad = 33\,\text{cm}. \end{aligned}$$

FOR INVESTIGATION

16.1 What is the area of the polygon $PQRSTUV$?

17. Amrita and Beatrix play a game in which each player starts with 10 counters. In each round of the game, one player wins and is given 3 counters; and her opponent has 1 counter removed. At the end of the game, Amrita and Beatrix have 40 counters and 16 counters respectively.
How many rounds of the game did Amrita win?

A 10 B 11 C 12 D 13 E 14

SOLUTION **C**

In each round Amrita and Beatrix between them have a net gain of 2 counters. They begin with $10 + 10 = 20$ counters and end with $40 + 16 = 56$ counters. This represents a gain of $56 - 20 = 36$ counters. It follows that there were $36 \div 2 = 18$ rounds.

Suppose that Amrita wins x of these rounds. It follows that she loses $18 - x$ rounds. Hence she gains $3x$ counters in the rounds that she wins, and loses $18 - x$ counters in the rounds that she loses.

Amrita begins with 10 counters and ends with 40 counters. Therefore

$$10 + 3x - (18 - x) = 40.$$

This equation may be rearranged to give

$$4x = 48,$$

and therefore

$$x = 12.$$

Therefore Amrita wins 12 rounds.

FOR INVESTIGATION

17.1 Amrita and Beatrix play a chess match.

If a game is won, the winner gains 3 points and the loser gains 0 points.

If a game is drawn each player gains 1 point.

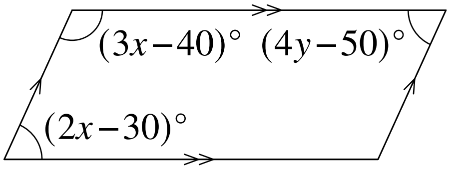
Amrita wins twice as many games as Beatrix and ends with 24 points.

Beatrix ends with 15 points.

How many games were there in the match, and how many of these games did Amrita win, how many did she draw, and how many did she lose?

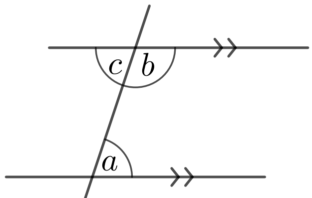
18. The diagram shows a parallelogram. What is the value of y?

A 22 B 24 C 25 D 28 E 30



SOLUTION **E**

When parallel lines are cut by a transversal, the angles marked a and b in the diagram on the right are called *included angles*. (Some people use the alternative name *C-angles*.)



In the solution to this question we use the fact that with a pair of parallel lines the sum of the included angles is 180°.

This follows from the facts that

$$a = c \qquad \text{(alternate angles)}$$

and

$$c + b = 180° \qquad \text{(angles on a line)}.$$

Hence

$$a + b = 180°.$$

In the parallelogram of the question there are two pairs of included angles. Therefore, using the above fact about included angles, we have

$$(3x - 40) + (2x - 30) = 180 \qquad (1)$$

and

$$(3x - 40) + (4y - 50) = 180. \qquad (2)$$

By equation (1),

$$5x = 250$$

and therefore

$$x = 50.$$

Hence, by equation (2),

$$110 + (4y - 50) = 180.$$

It follows that

$$4y = 120.$$

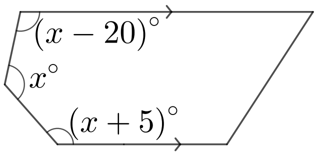
Therefore

$$y = 30.$$

FOR INVESTIGATION

18.1 Two sides of the pentagon are parallel, as shown in the diagram.

What is the value of x?



19. At the start of the day I had three times as many apples as pears. By the end of the day, after eating five apples but no pears, I had twice as many pears as apples.
How many pieces of fruit did I have at the start of the day?

A 4 B 8 C 12 D 16 E 20

SOLUTION **B**

Let p be the number of pears that I had at the start of the day. Since I had three times as many apples as pears, I began the day with $3p$ apples. It follows that I ended the day with $3p - 5$ apples and p pears.

Since I ended the day with twice as many pears as apples

$$p = 2(3p - 5),$$

that is

$$p = 6p - 10,$$

and hence

$$5p = 10.$$

We deduce that $p = 2$. It follows that I started the day with 2 pears and 6 apples. This makes a total of 8 pieces of fruit.

FOR INVESTIGATION

19.1 Another day I started with twice as many apples as pears. During the day I ate 9 apples and 8 pears. I ended the day with three times as many apples as pears.

How many apples did I have at the beginning of the day?

20. During a particularly troublesome lesson, the following conversation occurs:
Pam: "I always tell the truth." Quentin: "Pam is lying."
Roger: "Both Pam and Quentin are lying." Susan: "Everyone is lying."
Terry: "Everyone is telling the truth."

How many people are telling the truth?

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

If Pam is telling the truth, then Quentin is lying. So they cannot both be telling the truth.

If Quentin is lying then Pam is telling the truth. So they cannot both be lying.

Therefore one of Pam and Quentin is telling the truth and one is lying.

It follows that Roger, Susan and Terry are all lying.

So just one person was telling the truth.

21. Two lists of numbers are as shown below.

List S:	3	5	8	11	13	14
List T:	2	5	6	10	12	13

Jenny decided she would move one number from List S to List T and one number from List T to List S so that the sum of the numbers in the new List S is equal to the sum of the numbers in the new List T.

In how many ways could she do this?

A 1 B 2 C 3 D 4 E 5

SOLUTION **C**

The sum of the numbers in List S is $3 + 5 + 8 + 11 + 13 + 14 = 54$. The sum of the numbers in List T is $2 + 5 + 6 + 10 + 12 + 13 = 48$.

Now $54 + 48 = 102$ and $102 \div 2 = 51$. Therefore if the totals of the new lists are equal, each of the new lists will have total 51.

Therefore to make the sum of the numbers in the two lists equal, Jenny needs to reduce the sum of the numbers in List S by 3 and increase the sum of the numbers in List T by 3.

To achieve this Jenny needs to swap a number from List S with a number which is 3 less from List T.

Therefore the pairs of numbers that Jenny could swap are $(5, 2)$ or $(8, 5)$ or $(13, 10)$. No other swap would work.

Therefore the number of ways in which Jenny could make the sum of the numbers in the two new lists equal is 3.

FOR INVESTIGATION

21.1 In each of the following cases Jenny wishes to swap a number in List S with a number in List T so the sum of the numbers in the new List S is the same as the sum of the numbers in the new List T.

In each case, in how many ways could she do this?

(a)

List S:	3	5	8	11	13	14
List T:	4	6	9	10	12	15

(b)

List S:	4	7	9	13	15	17
List T:	2	5	8	12	18	21

21.2 In the following case Jenny wishes to swap a number in List S with a number in List T so that the sum of the numbers in the new list S is *twice* the sum of the numbers in the new List T.

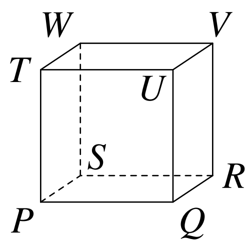
In how many ways could she do this?

List S:	8	9	10	13	14	15
List T:	1	2	3	4	5	6

22. A triangular pyramid with vertices T, U, V and Q is removed from the solid cube shown.

How many edges does the resulting solid have?

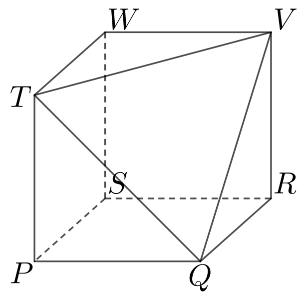
A 4 B 6 C 8 D 10 E 12



Solution **E**

When the triangular pyramid with vertices T, U, V and Q is removed, the three edges TU, UV and UQ of the cube are removed, and the resulting solid has three new edges TQ, QV and TV.

Therefore the new solid has the same number of edges as the cube. That is, the resulting solid has 12 edges.



For investigation

22.1 There is a famous formula, due to the great Swiss mathematician Leonhard Euler (1707-1783) which relates the number of vertices, edges and faces of shapes such as the cube of this question and the solid that results from removing the triangular pyramid.

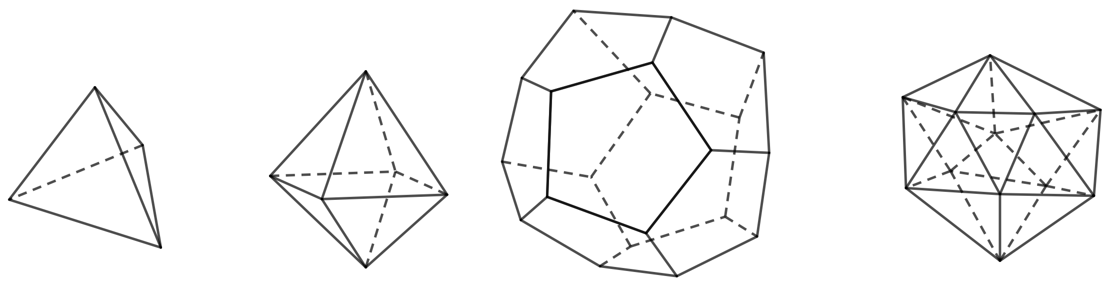
We let V be the number of vertices that such a shape has, E the number of edges, and F the number of faces. Euler's formula is

$$V - E + F = 2.$$

For the cube we have $V = 8$, $E = 12$ and $F = 6$. It can be seen that these values satisfy Euler's formula.

(a) Show that Euler's formula holds also for the shape that results in this question when the triangular prism is removed from the cube.

(b) The cube is an example of a *Platonic solid*. The Platonic solids are the polyhedra whose faces are congruent regular polygons, with the same number of faces meeting at each vertex. The other four Platonic solids are the regular tetrahedron, the regular octahedron, the regular dodecahedron, and the regular icosahedron. Check that Euler's formula holds for all of these Platonic solids.



tetrahedron octahedron dodecahedron icosahedron

(c) Check that Euler's formula holds for some other examples of polyhedra.

23. The price of a train ticket increased by 5% and then decreased by 20% in a special offer. It was now £4 less expensive than its original price. What was the original price of the ticket?

A £8.60 B £13 C £20.40 D £25 E £26.40

Solution **D**

Let the original price of the ticket be £p.

When the price of the ticket is increased by 5%, the price is multiplied by $\frac{105}{100}$. When the price is decreased by 20%, the price is multiplied by $\frac{80}{100}$.

Therefore the new price is £$\left(\frac{80}{100} \times \frac{105}{100} \times p\right)$.

Now $\frac{80}{100} \times \frac{105}{100} = \frac{4}{5} \times \frac{21}{20} = \frac{21}{25}$.

Therefore the new price is £$\left(\frac{21}{25}p\right)$.

Since the new price is £4 less than the original price,

$$\frac{21}{25}p = p - 4.$$

It follows that

$$\frac{4}{25}p = 4,$$

and hence

$$p = 25.$$

Hence hence the original price of the ticket was £25.

For investigation

23.1 A train ticket cost £23.80. The price of the ticket was increased by 20%. Then, in a special offer, the price was reduced so that the ticket now cost £2.38 less than the original price.

By what percentage was the price reduced in the special offer?

23.2 The cost of a train ticket was increased by 20%. After a protest the cost was reduced so that the ticket cost just 5% more than the original price.

By what percentage was the price reduced after the protest?

24. Flori's Flower shop contains fewer than 150 flowers. All the flowers are purple, yellow, red or white. The ratio of purple flowers to yellow flowers is 1 : 2, the ratio of yellow flowers to red flowers is 3 : 4 and the ratio of red flowers to white flowers is 5 : 6.
How many flowers are there in Flori's shop?

A 133 B 136 C 139 D 142 E 145

SOLUTION **A**

Let the numbers of purple, yellow, red and white flowers be p, y, r and w, respectively.

Since $p : y = 1 : 2$, we have $\frac{p}{y} = \frac{1}{2}$. Therefore $y = 2p$.

Since $y : r = 3 : 4$, we have $\frac{y}{r} = \frac{3}{4}$. Therefore $r = \frac{4}{3}y = \frac{4}{3}(2p) = \frac{8}{3}p$.

Since $r : w = 5 : 6$, we have $\frac{r}{w} = \frac{5}{6}$. Therefore $w = \frac{6}{5}r = \frac{6}{5}\left(\frac{8}{3}p\right) = \frac{48}{15}p$.

It follows that the total number of flowers is

$$\begin{aligned} p + y + r + w &= p + 2p + \frac{8}{3}p + \frac{48}{15}p \\ &= \left(1 + 2 + \frac{8}{3} + \frac{48}{15}\right)p \\ &= \left(\frac{15}{15} + \frac{30}{15} + \frac{40}{15} + \frac{48}{15}\right)p \\ &= \frac{133}{15}p. \end{aligned}$$

This number is a positive integer. For $\frac{133}{15}p$ to be a positive integer p needs to be a multiple of 15. That is, $p = 15k$, where k is some positive integer.

When $p = 15k$, we have $\frac{133}{15}p = 133k$.

Since the flower shop contains fewer than 150 flowers, $k = 1$. This makes the total number of flowers 133.

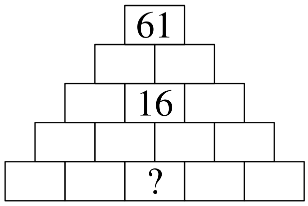
FOR INVESTIGATION

24.1 Fleur's Flower shop contains fewer than 300 pot plants. They are all azaleas, begonias, cyclamen or dracaenas. The ratio of azaleas to begonias is 1 : 3, the ratio of begonias to cyclamen is 2 : 5, and the ratio of cyclamen to dracaenas is 4 : 7.

How many pot plants are there in Fleur's shop?

25. In the number pyramid, each cell above the bottom row contains the sum of the numbers in the two cells immediately below it. The sum of the numbers in the bottom row is 17. What is the central number of the bottom row?

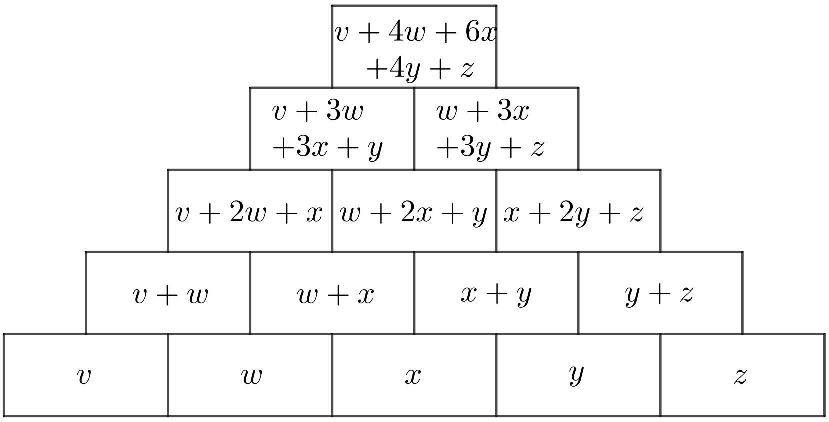
A 2 B 3 C 4 D 5 E 6



Solution **C**

Let the numbers in the bottom row of the pyramid be v, w, x, y and z, in this order.

Because each other number in the pyramid is the sum of the numbers in the two cells immediately below it, these numbers are as shown in the diagram.



From what we are told in the question

$$v + w + x + y + z = 17, \qquad (1)$$
$$w + 2x + y = 16, \qquad (2)$$
$$\text{and} \qquad v + 4w + 6x + 4y + z = 61. \qquad (3)$$

Note: We have five unknowns but only three equations. These equations do not have a unique solution. However, we can use the equations to find the value of x.

By (1) and (3), $(v + 4w + 6x + 4y + z) - (v + w + x + y + z) = 61 - 17,$

that is $3w + 5x + 3y = 44. \qquad (4)$

Now, by (2) and (4), $3(w + 2x + y) - (3w + 5x + 3y) = 3 \times 16 - 44,$

that is $x = 4.$

Note: In the context of the JMC, we can stop here. For a full solution you would need to show that the number pyramid may be completed with $x = 4$. Try to do this and also see Problem 25.1.

For investigation

25.1 Show that it is possible to complete the number pyramid of this question with five different positive integers with sum 17 in the bottom row. In how many ways can this be done?

25.2 Consider a number pyramid with 6 rows, with the numbers u, v, w, x, y and z, in this order, in the bottom row. Find an expression in terms of u, v, w, x, y, z for the number in the top cell? What is this number when $u = 25$, $v = 40$, $w = 72$, $x = 84$, $y = 42$ and $z = 27$?

25.3 What do you notice about the coefficients that appear in the expressions in the top four rows of the number pyramid in the diagram of the solution above?